

STABILITY AND COMPACTNESS FOR COMPLETE f -MINIMAL SURFACES

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ABSTRACT. Let $(M, \bar{g}, e^{-f} d\mu)$ be a complete metric measure space with Bakry-Émery Ricci curvature bounded below by a positive constant. We prove that, in M , there is no complete two-sided L_f -stable immersed f -minimal hypersurface with finite weighted volume. Further, if M is a 3-manifold, we prove a smooth compactness theorem for the space of complete embedded f -minimal surfaces in M with the uniform upper bounds of genus and weighted volume, which generalizes the compactness theorem for complete self-shrinkers in \mathbb{R}^3 by Colding-Minicozzi.

1. INTRODUCTION

Recall that a self-shrinker (for mean curvature flow in \mathbb{R}^{n+1}) is a hypersurface Σ immersed in the Euclidean space $(\mathbb{R}^{n+1}, g_{can})$ satisfying that

$$H = \frac{1}{2} \langle x, \nu \rangle,$$

where x is the position vector in \mathbb{R}^{n+1} , ν is the unit normal at x , and H is the mean curvature of Σ at x . Self-shrinkers play an important role in the study of singularity of mean curvature flow and have been studied by many people in recent years. We refer to [4], [5] and the references therein. In particular, Colding-Minicozzi [4] proved the following compactness theorem for self-shrinkers in \mathbb{R}^3 .

Theorem 1. [4] *Given an integer $g \geq 0$ and a constant $V > 0$, the space $\mathcal{S}(g, V)$ of smooth complete embedded self-shrinkers $\Sigma \subset \mathbb{R}^3$ with*

- *genus at most g ,*
- *$\partial \Sigma = \emptyset$,*
- *$\text{Area}(B_R(x_0) \cap \Sigma) \leq VR^2$ for all $x_0 \in \mathbb{R}^3$ and $R > 0$*

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is compact.

Namely, any sequence of these has a subsequence that converges in the topology of C^m convergence on compact subsets for any $m \geq 2$.

In this paper, we extend Theorem 1 to the space of complete embedded f -minimal surfaces in a 3-manifold. A hypersurface Σ immersed in a Riemannian manifold (M, \bar{g}) is called a f -minimal hypersurface if its mean curvature H satisfies that, for any $p \in \Sigma$,

$$H = \langle \bar{\nabla} f, \nu \rangle,$$

where f is a smooth function defined on M , and $\bar{\nabla} f$ denotes the gradient of f on M . Here are some examples of f -minimal hypersurfaces:

- $f \equiv C$, an f -minimal hypersurface is just a minimal hypersurface;
- self-shrinker Σ in \mathbb{R}^{n+1} . $f = \frac{|x|^2}{4}$;
- Let (M, \bar{g}, f) be a shrinking gradient Ricci solitons, i.e. after a normalization, (M, \bar{g}, f) satisfies the equation $\bar{\text{Ric}} + \bar{\nabla}^2 f = \frac{1}{2}\bar{g}$ or equivalently the Bakry-Émery Ricci curvature $\bar{\text{Ric}}_f := \bar{\text{Ric}} + \bar{\nabla}^2 f = \frac{1}{2}$. We may consider f -minimal hypersurfaces in (M, \bar{g}, f) . In particular, the previous example: a self-shrinker Σ in \mathbb{R}^{n+1} is f -minimal in Gauss shrinking soliton $(\mathbb{R}^{n+1}, g_{can}, \frac{|x|^2}{4})$;
- $M = \mathbb{H}^{n+1}(-1)$, the hyperbolic space. Let r denote the distance function from a fixed point $p \in M$ and $f(x) = nar^2(x)$, where $a > 0$ is a constant. Now $\bar{\text{Ric}}_f \geq n(2a - 1)$. The geodesic sphere of radius r centered at p is an f -minimal hypersurface if the radius r satisfies $2ar = \coth r$.

An f -minimal hypersurface Σ can be viewed in two ways. One is that Σ is f -minimal if and only if Σ is a critical point of the weighted volume functional $e^{-f}d\sigma$, where $d\sigma$ is the volume element of Σ . The other one is that Σ is f -minimal if and only if Σ is minimal in the new conformal metric $\tilde{g} = e^{-\frac{2f}{n}}\bar{g}$ (see Section 2 and Appendix). f -minimal hypersurfaces have been studied before, even more general stationary hypersurfaces for parametric elliptic functionals, see for instance the work of White [14] and Colding-Minicozzi [7].

We prove the following compactness result:

Theorem 2. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space and $\overline{\text{Ric}}_f \geq k$, where k is a positive constant. Given an integer $g \geq 0$ and a constant $V > 0$, the space $S_{g,V}$ of smooth complete embedded f -minimal surfaces $\Sigma \subset M$ with*

- *genus at most g ,*
- *$\partial\Sigma = \emptyset$,*
- *$\int_{\Sigma} e^{-f} d\sigma \leq V$*

is compact in the C^m topology, for any $m \geq 2$. Namely, any sequence of $S_{g,V}$ has a subsequence that converges in the C^m topology on compact subsets to a surface in $S_{D,g}$, for any $m \geq 2$.

Since the existence of the uniform scale-invariant area bound is equivalent to the existence of the uniform bound of the weighted area for self-shrinkers (see Remark 1 in Section 5), Theorem 2 implies Theorem 1. Also, in [2], we will apply Theorem 2 to obtain a compactness theorem for the space of closed embedded f -minimal surfaces with the upper bounds of genus and diameter.

To prove Theorem 2, we need to prove a non-existence result on L_f -stable f -minimal hypersurfaces, which is of independent interest.

Theorem 3. *Let $(M^{n+1}, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{\text{Ric}}_f \geq k$, where k is positive constant. Then there is no complete two-sided L_f -stable f -minimal hypersurface Σ immersed in (M, \bar{g}) without boundary and with finite weighted volume (i.e. $\int_{\Sigma} e^{-f} d\sigma < \infty$), where $d\sigma$ denotes the volume element on Σ determined by the induced metric from (M, \bar{g}) .*

Here we explain briefly the meaning of L_f stability. For an f -minimal hypersurface Σ , L_f operator is

$$L_f = \Delta_f + |A|^2 + \overline{\text{Ric}}_f(\nu, \nu),$$

where $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ is the weighted Laplacian on Σ .

Especially, for self-shrinkers, it is so-called L operator:

$$L = \Delta - |A|^2 - \frac{1}{2}\langle x, \nabla \cdot \rangle + \frac{1}{2}.$$

L_f -stability of Σ means that its weighted volume $\int_{\Sigma} e^{-f} d\sigma$ is locally minimal, that is, the second variation of its weighted volume is nonnegative for all compactly supported normal variation. We leave more details about the definition of L_f -stability and some of its properties in Section 2 and Appendix.

For self-shrinkers in \mathbb{R}^{n+1} , Colding-Minicozzi [6] proved that

Theorem 4. [6] *There are no L -stable smooth complete self-shrinkers without boundary and with polynomial volume growth in \mathbb{R}^{n+1} .*

Since the first and third authors [3] of the present paper proved that for self-shrinkers, properness, the polynomial volume growth, and finite weighted volume are equivalent. Hence Theorem 3 implies Theorem 4.

In this paper, we discuss the relation among the properness, polynomial volume growth and finite weighted volume of f -minimal submanifolds (Propositions 3, 4 and 5). We obtain their equivalence when the ambient space (M, \bar{g}, f) is a shrinking gradient Ricci solitons, i.e. $\overline{\text{Ric}} + \overline{\nabla}^2 f = \frac{1}{2}\bar{g}$ with the condition $|\overline{\nabla} f|^2 \leq f$ (Corollary 1).

The rest of this paper is organized as follows: In Section 2 some definitions, notations and facts are given as a preliminary; In Section 3 we prove Propositions 3, 4 and 5; In Section 4 we prove Theorem 3; In Section 5 we prove Theorem 2; In Appendix we discuss some properties of L_f -stability for f -minimal submanifolds.

2. PRELIMINARIES

In general, a smooth metric measure space, denoted by $(M^m, \bar{g}, e^{-f} d\mu)$, is an m -dimensional Riemannian manifold (M^m, \bar{g}) together with a weighted volume form $e^{-f} d\mu$ on M , where f is a smooth function on M and $d\mu$ the volume element induced by the metric \bar{g} . In this paper, unless otherwise specified, we denote by a bar all quantities on (M, \bar{g}) , for instance by $\overline{\nabla}$ and $\overline{\text{Ric}}$, the Levi-Civita connection and the Ricci curvature tensor of (M, \bar{g})

respectively. For $(M, \bar{g}, e^{-f} d\mu)$, an important and natural tensor is the ∞ -Bakry-Émery Ricci curvature tensor $\overline{\text{Ric}}_f$ (for simplicity, Bakry-Émery Ricci curvature), which is defined by

$$\overline{\text{Ric}}_f := \overline{\text{Ric}} + \bar{\nabla}^2 f,$$

where $\bar{\nabla}^2 f$ is the Hessian of f on M . If f is constant, $\overline{\text{Ric}}_f$ is the Ricci curvature $\overline{\text{Ric}}$ on M respectively.

A Riemannian manifold with Bakry-Émery Ricci curvature bounded below by a positive constant has some properties similar to a Riemannian manifold with Ricci curvature bounded below by a positive constant. For instance, see the work of Wei-Wylie [13], Munteanu-Wang [11, 12] and the references therein. In this paper, we will use the following proposition by Morgan [10] (see also its proof in [13]).

Proposition 1. *If a complete smooth metric measure space $(M, \bar{g}, e^{-f} du)$ has $\overline{\text{Ric}}_f \geq k$, where k is a positive constant, then M has finite weighted volume (i.e. $\int_M e^{-f} d\mu < \infty$) and finite fundamental group.*

Now, let $i : \Sigma^n \rightarrow M^m, n < m$, be an n -dimensional smooth immersion. Then $i : (\Sigma^n, i^* \bar{g}) \rightarrow (M^m, \bar{g})$ is an isometric immersion with the induced metric $i^* \bar{g}$. For simplicity, we still denote $i^* \bar{g}$ by \bar{g} whenever there is no confusion. We will denote for instance by ∇ , Ric , Δ and $d\sigma$, the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the volume element of (Σ, \bar{g}) respectively.

The function f induces a weighted measure $e^{-f} d\sigma$ on Σ . Thus we have an induced smooth metric measure space $(\Sigma^n, \bar{g}, e^{-f} d\sigma)$.

The associated weighted Laplacian Δ_f on (Σ, \bar{g}) is defined by

$$\Delta_f u := \Delta u - \langle \nabla f, \nabla u \rangle.$$

The second order operator Δ_f is a self-adjoint operator on the space of square integrable functions on Σ with respect to the measure $e^{-f} d\sigma$ (however the Laplacian operator in general has no this properties).

The second fundamental form A of (Σ, \bar{g}) is defined by

$$A(X, Y) = (\bar{\nabla}_X Y)^\perp, \quad X, Y \in T_p \Sigma, p \in \Sigma,$$

where \perp denotes the projection to the normal bundle of Σ . The mean curvature vector \mathbf{H} of Σ is defined by $\mathbf{H} = \text{tr}A = \sum_{i=1}^n (\bar{\nabla}_{e_i} e_i)^\perp$.

Definition 1. *The weighted mean curvature vector of Σ with respect to the metric \bar{g} is defined by*

$$(1) \quad \mathbf{H}_f = \mathbf{H} + (\bar{\nabla} f)^\perp.$$

The immersed submanifold (Σ, \bar{g}) is called f -minimal if its weighted mean curvature vector \mathbf{H}_f vanishes identically, or equivalently if its mean curvature vector satisfies

$$(2) \quad \mathbf{H} = -(\bar{\nabla} f)^\perp.$$

Definition 2. *The weighted volume of (Σ, \bar{g}) is defined by*

$$(3) \quad V_f(\Sigma) := \int_{\Sigma} e^{-f} d\sigma.$$

It is well known that Σ is f -minimal if and only if Σ is a critical point of the weighted volume functional. Namely, it holds that

Proposition 2. *If T is a compactly supported variational field on Σ , then the first variation formula of the weighted volume of (Σ, \bar{g}) is given by*

$$(4) \quad \left. \frac{d}{dt} V_f(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \langle T^\perp, \mathbf{H}_f \rangle_{\bar{g}} e^{-f} d\sigma.$$

On the other hand, an f -minimal submanifold can be viewed as a minimal submanifold under a conformal metric. Precisely, define the new metric $\tilde{g} = e^{-\frac{2}{n}f} \bar{g}$ on M , which is conformal to \bar{g} . Then the immersion $i : \Sigma \rightarrow M$ induces a metric $i^* \tilde{g}$ on Σ from (M, \tilde{g}) . In the following, $i^* \tilde{g}$ is still denoted by \tilde{g} for simplicity. The volume of (Σ, \tilde{g}) is

$$(5) \quad \tilde{V}(\Sigma) := \int_{\Sigma} d\tilde{\sigma} = \int_{\Sigma} e^{-f} d\sigma = V_f(\Sigma).$$

Hence Proposition 2 and (5) imply that

$$(6) \quad \int_{\Sigma} \langle T^\perp, \tilde{\mathbf{H}} \rangle_{\tilde{g}} d\tilde{\sigma} = \int_{\Sigma} \langle T^\perp, \mathbf{H}_f \rangle_{\bar{g}} e^{-f} d\sigma,$$

where $d\tilde{\sigma} = e^{-f} d\sigma$ and $\tilde{\mathbf{H}}$ denote the volume element and the mean curvature vector of Σ with respect to the conformal metric \tilde{g} respectively.

Identity (6) implies that $\tilde{\mathbf{H}} = e^{\frac{2f}{n}} \mathbf{H}_f$ and (Σ, \bar{g}) is f -minimal in (M, \bar{g}) if and only if (Σ, \tilde{g}) is minimal in (M, \tilde{g}) .

Now suppose that Σ^n is a hypersurface immersed in M^{n+1} . Let $p \in \Sigma$ and ν a unit normal at p . The second fundamental form A and the mean curvature H of (Σ, \bar{g}) are as follows:

$$A : T_p \Sigma \rightarrow T_p \Sigma, A(X) = \bar{\nabla}_X \nu, X \in T_p \Sigma,$$

$$H = \text{tr} A = - \sum_{i=1}^n \langle \bar{\nabla}_{e_i} e_i, \nu \rangle.$$

Hence the mean curvature vector \mathbf{H} of (Σ, \bar{g}) satisfies $\mathbf{H} = -H\nu$. Define the weighted mean curvature H_f of (Σ, \bar{g}) by $\mathbf{H}_f := -H_f \nu$. Then

$$H_f = H - \langle \bar{\nabla} f, \nu \rangle.$$

Definition 3. A hypersurface Σ immersed in $(M^{n+1}, \bar{g}, e^{-f} d\mu)$ with the induced metric \bar{g} is called an f -minimal hypersurface if it satisfies

$$(7) \quad H = \langle \bar{\nabla} f, \nu \rangle.$$

For a hypersurface (Σ, \bar{g}) , the L_f operator is defined by

$$L_f := \Delta_f + |A|^2 + \overline{\text{Ric}}_f(\nu, \nu),$$

where $|A|^2$ denotes the square of the norm of the second fundamental form A of Σ .

The L_f -stability of Σ is defined as follows:

Definition 4. A two-sided f -minimal hypersurface Σ is said to be L_f -stable if for any compactly supported smooth function $\varphi \in C_0^\infty(\Sigma)$, it holds that

$$(8) \quad - \int_{\Sigma} \varphi L_f \varphi e^{-f} d\sigma = \int_{\Sigma} [|\nabla \varphi|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu)) \varphi^2] e^{-f} d\sigma \geq 0.$$

It is known that an f -minimal hypersurface (Σ, \bar{g}) is L_f -stable if and only if (Σ, \tilde{g}) is stable as a minimal surface with respect to the conformal metric $\tilde{g} = e^{-f} \bar{g}$. See more details in Appendix of this paper.

In this paper, for closed hypersurfaces, we choose ν to be the outer unit normal.

3. PROPERNESS, POLYNOMIAL VOLUME GROWTH AND FINITE WEIGHTED VOLUME OF f -MINIMAL HYPERSURFACES

In [3], the first and third authors of the present paper proved that the finite weighted volume of a self-shrinker Σ^n immersed in \mathbb{R}^m implies it is properly immersed. In [9], Ding-Xin proved that a properly immersed self-shrinker must have the Euclidean volume growth. Combining these two results, it was proved [3] that for immersed self-shrinkers, properness, polynomial volume growth and finite weighted volume are equivalent.

In this section we study the relation among the properness, polynomial volume growth and finite weighted volume of f -minimal submanifolds, some of them will be used later in this paper.

If Σ is an n -dimensional submanifold in a complete manifold $M^m, n < m$, Σ is said to have polynomial volume growth if, for a $p \in M$ fixed, there exist constants C and d so that for all $r \geq 1$,

$$(9) \quad \text{Vol}(B_r^M(p) \cap \Sigma) \leq Cr^d,$$

where $B_r^M(p)$ is the extrinsic ball of radius r centered at p , $\text{Vol}(B_r^M(p))$ denotes the volume of $B_r^M(p) \cap \Sigma$. When $d = n$ in (9), Σ is said to be of Euclidean volume growth.

Before proving the following Proposition 3, we recall an estimate implied by the Hessian comparison theorem (cf, for instance, [6] Lemma 7.1).

Lemma 1. *Let (M, \bar{g}) be a complete Riemannian manifold with bounded geometry, that is, M has sectional curvature bounded by k ($|K_M| \leq k$), and injectivity radius bounded below by $i_0 > 0$. Then the distance function $r(x)$ satisfies*

$$|\bar{\nabla}^2 r(V, V) - \frac{1}{r}|V - \langle V, \bar{\nabla} r \rangle \bar{\nabla} r|^2| \leq \sqrt{k},$$

for $r < \min\{i_0, \frac{1}{\sqrt{k}}\}$ and any unit vector $V \in T_x \Sigma$.

Using this estimate we will prove that

Proposition 3. *Let Σ^n be a complete noncompact f -minimal submanifold immersed in a complete Riemannian manifold M^m . If Σ has finite weighted area, then Σ is properly immersed in M .*

Proof. We argue by contradiction. Since the argument is local, we may assume that (M, g) has bounded geometry. Suppose that Σ is not properly immersed. Then there exist a number $2R < \min\{i_0, \frac{1}{\sqrt{k}}\}$ and $o \in M$ so that $\overline{B}_R^M(o) \cap \Sigma$ is not compact in Σ , where $B_R^M(o)$ denotes an extrinsic ball of radius R centered at o . Then for any $a > 0$, there is a sequence $\{p_k\}$ of points in $B_R^M(o) \cap \Sigma$ with $\text{dist}_\Sigma(p_k, p_j) \geq a > 0$ for any $k \neq j$. So $B_{\frac{a}{2}}^\Sigma(p_k) \cap B_{\frac{a}{2}}^\Sigma(p_j) = \emptyset$ for any $k \neq j$. Choose $a < 2R$. Then $B_{\frac{a}{2}}^\Sigma(p_j) \subset B_{2R}^M(o)$. If $p \in B_{\frac{a}{2}}^\Sigma(p_j)$, the extrinsic distance function $r_j(p) = \text{dist}_M(p, p_j)$ from p_j satisfies,

$$\begin{aligned} \Delta r_j &= \sum_{i=1}^n \overline{\nabla}^2 r_j(e_i, e_i) + \langle \mathbf{H}, \overline{\nabla} r_j \rangle \\ &\geq \frac{n}{r_j} - \frac{1}{r_j} |\nabla r_j|^2 - n\sqrt{k} - \langle \overline{\nabla} f^\perp, \overline{\nabla} r_j \rangle \\ &\geq \frac{n}{r_j} - \frac{1}{r_j} |\nabla r_j|^2 - c, \end{aligned}$$

where $c = n\sqrt{k} + \sup_{B_{2R}^M(o)} |\overline{\nabla} f|$. Hence

$$\Delta r_j^2 \geq 2n - 2cr_j.$$

Choosing $a \leq \min\{\frac{n}{2c}, 2R\}$, we have for $0 < \mu \leq \frac{a}{2}$

$$\begin{aligned} (10) \quad \int_{B_\mu^\Sigma(p_j)} (2n - 2cr_j) d\sigma &\leq \int_{B_\mu^\Sigma(p_j)} \Delta_\Sigma r_j^2 d\sigma \\ &= \int_{\partial B_\mu^\Sigma(p_j)} \langle \nabla^\Sigma r_j^2, \nu \rangle d\sigma \\ &\leq 2\mu A(\mu), \end{aligned}$$

where ν denotes the outward unit normal vector of $\partial B_\mu^\Sigma(p_j)$ and $A(\mu)$ denotes the area of $\partial B_\mu^\Sigma(p_j)$. Using co-area formula in (10), we have

$$(11) \quad \int_0^\mu (n - cs) A(s) ds \leq \int_0^\mu \int_{d_\Sigma(p, p_j)=s} (n - cr_j) d\sigma \leq \mu A(\mu).$$

This implies

$$(n - c\mu)V(\mu) \leq V'(\mu),$$

where $V(\mu)$ denotes the volume of $B_\mu^\Sigma(p_j)$. So

$$(12) \quad \frac{V'(\mu)}{V(\mu)} \geq \frac{n}{\mu} - c$$

Integrating (12) from $\varepsilon > 0$ to μ , we have

$$\frac{V(\mu)}{V(\varepsilon)} \geq \left(\frac{\mu}{\varepsilon}\right)^n e^{-c(\mu-\varepsilon)}$$

Since $\lim_{s \rightarrow 0^+} \frac{V(s)}{s^n} = \omega_n$,

$$(13) \quad V(\mu) \geq \omega_n \mu^n e^{-c\mu}.$$

Thus we conclude

$$\int_{\Sigma} e^{-f} d\sigma \geq \sum_{j=1}^{\infty} \int_{B_{\frac{\Sigma}{2}}(p_j)} e^{-f} d\sigma \geq \inf_{B_{2R}^M(o)} (e^{-f}) \sum_{j=1}^{\infty} \int_{B_{\frac{\Sigma}{2}}(p_j)} d\sigma = \infty.$$

This contradicts with the assumption of the finite weighted volume of Σ . \square

Proposition 4. *Let $(M^m, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{Ric}_f = k$, where k is a positive constant. Assume that f is a convex function. If Σ^n is a complete noncompact properly immersed f -minimal submanifold, then Σ has finite weighted volume and Euclidean (hence polynomial) volume growth.*

Proof. Since (M, \bar{g}, f) is a gradient shrinking Ricci soliton, it is well-known that, by a scaling of the metric \bar{g} and a translating of f , still denoted by \bar{g} and f , we may normalize the metric so that $k = \frac{1}{2}$ and the following identities hold:

$$\begin{aligned} \bar{R} + |\bar{\nabla} f|^2 - f &= 0, \\ \bar{R} + \bar{\Delta} f &= \frac{m}{2}, \\ \bar{R} &\geq 0. \end{aligned}$$

From these equations, we have that

$$\bar{\Delta} f - |\bar{\nabla} f|^2 + f = \frac{m}{2}, \quad \text{and} \quad |\bar{\nabla} f|^2 \leq f.$$

It was proved by Cao and the third author [1] that there is a positive constant c so that

$$(14) \quad \frac{1}{4}(r(x) - c)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c)^2.$$

for any $x \in M$ with $r(x) = \text{dist}_M(p, x) \geq r_0$, where p is a fixed point in M and c, r_0 are positive constants that depend only of m and $f(p)$.

By (14), we know that f is a proper function on M . Since Σ is properly immersed in M and f is proper in M , $f|_{\Sigma}$ is also a proper smooth function on Σ . Note that with the scaling metric and translating f , Σ is still f -minimal. Hence

$$\begin{aligned} \Delta f - |\nabla f|^2 + f &= (\overline{\Delta} f - \sum_{\alpha=n+1}^m f_{\alpha\alpha} - |\overline{\nabla} f^\perp|^2) - |\overline{\nabla} f^\top|^2 + f \\ &= \overline{\Delta} f - |\overline{\nabla} f|^2 + f - \sum_{\alpha=n+1}^m f_{\alpha\alpha} \\ &\leq \frac{m}{2}. \end{aligned}$$

Also we have

$$|\nabla f|^2 = |\overline{\nabla} f^\top|^2 \leq |\overline{\nabla} f|^2 \leq f.$$

By Theorem 1.1 of [3], Σ has finite weighted volume and the Euclidean volume growth of the sub-level set of f with the respect to the scaling metric and the translating f , and hence with the respect to the original metric and f . Moreover, by the estimate (14), we have that Σ has the Euclidean volume growth.

□

We prove the following

Proposition 5. *Let $(M^m, \overline{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{\text{Ric}}_f \geq k$, where k is a positive constant. Assume that $|\overline{\nabla} f|^2 \leq 2kf$. If Σ^n is a complete submanifold (not necessarily f -minimal) with polynomial area growth, then Σ has finite weighted volume.*

Proof. By a scaling of the metric, we may assume that $k = \frac{1}{2}$. The proof follows from the estimate of f . Munteanu-Wang [11] extended the estimate (14) to $(M^m, \overline{g}, e^{-f} d\mu)$ with $\overline{\text{Ric}}_f \geq \frac{1}{2}$ and $|\overline{\nabla} f|^2 \leq f$. Combining the assumption that Σ has polynomial volume growth with the estimative (14),

we have

$$\begin{aligned}
\int_{\Sigma} e^{-f} d\sigma &= \int_{\Sigma \cap B_{r_0}^M(p)} e^{-f} d\sigma + \sum_{i=0}^{\infty} \int_{\Sigma \cap (B_{r_0+i+1}^M(p) \setminus B_{r_0+i}^M(p))} e^{-f} d\sigma, \\
&\leq C_1 \text{Vol}(\Sigma \cap B_{r_0}^M(p)) + C \sum_{i=0}^{\infty} e^{-\frac{1}{4}(r_0+i-c)^2} \text{Vol}(\Sigma \cap B_{r_0+i+1}^M(p)), \\
&\leq C \left[r_0^d + \sum_{i=0}^{\infty} e^{-\frac{1}{4}(r_0+i-c)^2} (r_0+i+1)^d \right], \\
&< \infty.
\end{aligned}$$

□

By Propositions 3, 4 and 5, we have the following

Corollary 1. *Let (M^m, \bar{g}, f) be a complete shrinking gradient Ricci soliton with $\bar{\text{Ric}}_f \geq \frac{1}{2}$. Assume that $|\bar{\nabla} f|^2 \leq f$. If Σ is a complete f -minimal submanifold immersed in M , then for Σ the properness, polynomial volume growth, and finite weighted volume are equivalent.*

4. NON-EXISTENCE OF L_f STABLE f -MINIMAL HYPERSURFACES

In this section, we prove Theorem 3, which is a key to prove the compactness theorem in Section 5.

Theorem 5. *(Theorem 3) Let $(M, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\bar{\text{Ric}}_f \geq k$, where k is a positive constant. Then there is no two-sided L_f -stable complete f -minimal hypersurface Σ immersed in (M, g) without boundary and with finite weighted volume (i.e. $\int_{\Sigma} e^{-f} d\sigma < \infty$).*

Proof. We argue by contradiction. Suppose that Σ is an L_f -stable complete f -minimal hypersurface immersed in (M, g) without boundary and with finite weighted volume. Recall that a two-sided hypersurface Σ is L_f -stable if the following inequality holds that, for any compactly supported smooth function $\varphi \in \mathcal{C}_0^\infty(\Sigma)$,

$$(15) \quad \int_{\Sigma} [|\nabla \varphi|^2 - (|A|^2 + \bar{\text{Ric}}_f(\nu, \nu))] e^{-f} d\sigma \geq 0.$$

Observe that any closed hypersurface cannot be L_f -stable. This is because that: the assumption $\bar{\text{Ric}}_f \geq k > 0$ implies that (15) cannot hold for $\varphi \equiv c$ on Σ . Hence, Σ must be noncompact.

Let η be a nonnegative smooth function on $[0, \infty)$ satisfying

$$\eta(s) = \begin{cases} 1 & \text{if } s \in [0, 1) \\ 0 & \text{if } s \in [2, \infty) \end{cases}$$

and $|\eta'| \leq 2$.

Fix a point $p \in \Sigma$ and let $r(x) = \text{dist}_\Sigma(p, x)$ denote the (intrinsic) distance function in Σ . Define a sequence of functions $\varphi_j(x) = \eta(\frac{r(x)}{j})$, $j \geq 1$. Then $|\nabla \varphi_j|^2 \leq 1$ for $j \geq 2$. Substituting φ_j for φ in (15):

$$\begin{aligned} & \int_\Sigma \left(|\nabla \varphi_j|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu)) \varphi_j^2 \right) e^{-f} d\sigma \\ & \leq \int_\Sigma (|\nabla \varphi_j|^2 - k \varphi_j^2) e^{-f} d\sigma \\ & = \int_{B_{2j}^\Sigma(p) \setminus B_j^\Sigma(p)} |\nabla \varphi_j|^2 e^{-f} d\sigma - \int_{B_{2j}^\Sigma(p)} k \varphi_j^2 e^{-f} d\sigma \\ & \leq \int_{B_{2j}^\Sigma(p) \setminus B_j^\Sigma(p)} e^{-f} d\sigma - k \int_{B_{2j}^\Sigma(p)} e^{-f} d\sigma \\ & \leq \int_{B_{2j}^\Sigma(p) \setminus B_j^\Sigma(p)} e^{-f} d\sigma - k \int_{B_2^\Sigma(p)} e^{-f} d\sigma, \end{aligned}$$

where $B_j^\Sigma(p)$ is the intrinsic geodesic ball in M of radius j centered at p . Since Σ has finite weighted volume, we have, when $j \rightarrow \infty$

$$\int_{B_{2j}^\Sigma(p) \setminus B_j^\Sigma(p)} e^{-f} d\sigma \rightarrow 0.$$

Choosing j large enough, we have that φ_j satisfies

$$\int_\Sigma (|\nabla \varphi_j|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu)) \varphi_j^2) e^{-f} d\sigma < -\frac{k}{2} \int_{B_2^\Sigma(p)} e^{-f} d\sigma < 0.$$

This contradicts that Σ is L_f -stable. □

5. COMPACTNESS OF COMPLETE f -MINIMAL SURFACES

Before proving Theorem 2, we give some facts.

Wei-Wylie ([13] Theorem 7.3) used mean curvature comparison theorem to give a distance estimate for two compact hypersurfaces Σ_1 and Σ_2 in a smooth metric measure space $(M, \bar{g}, e^{-f} d\mu)$ with $\overline{\text{Ric}}_f \geq k$, where k is a positive constant. Observe that for two complete hypersurfaces Σ_1 and Σ_2 if at least one of them is compact, there is a minimal geodesic joining Σ_1

and Σ_2 and realizing their distance. Hence the proof of Theorem 7.3 [13] can be applied to obtain the following

Proposition 6. *Let $(M, \bar{g}, e^{-f} d\mu)$ be an $(n+1)$ -dimensional smooth metric measure space with $\overline{\text{Ric}}_f \geq k$, where k is a positive constant. If Σ_1 and Σ_2 are two complete immersed hypersurfaces, at least one of them is compact, then the distance $d(\Sigma_1, \Sigma_2)$ satisfies*

$$(16) \quad d(\Sigma_1, \Sigma_2) \leq \frac{1}{k} (\max_{x \in \Sigma_1} |H_f^{\Sigma_1}(x)| + \max_{x \in \Sigma_2} |H_f^{\Sigma_2}(x)|).$$

Corollary 2. *Let $(M, \bar{g}, e^{-f} d\mu)$ be as in Proposition 6. Then there is a closed ball B^M of M satisfies that any complete immersed f -minimal hypersurface Σ must intersect it.*

Proof. Fix $p \in M$ and a geodesic sphere $B_r^M(p)$ of M . By Proposition 6,

$$d(B_r^M(p), \Sigma) \leq \frac{1}{2k} \max_x |H_f^{B_r^M(p)}(x)| = C,$$

where C is independent on Σ . Therefore there is a closed ball B^M of M with radius big enough so that any Σ must intersect it.

□

We need the following fact:

Proposition 7. *Let M be a simply connected Riemannian manifold. If an f -minimal hypersurface Σ is complete, not necessarily connected, properly embedded, and has no boundary, then every component of Σ separates M into two components and thus is two-sided. Therefore Σ has a globally defined unit normal.*

Proof. Suppose Σ_j is a component of Σ . By contrary, if $M \setminus \Sigma_j$ has one component. Since Σ is a properly embedded f -minimal hypersurface, for any $p \in \Sigma_j$, there is a neighborhood W of p in M so that $W \cap \Sigma_j = W \cap \Sigma$ only has one piece (i.e. it is a graph above a connected domain in the tangent plane of p). Thus we have a simply closed curve γ passing p , transversal to Σ_j at p , and $\Sigma_j \cap \gamma = p$. Since M is simply connected, we have a disk D with the boundary γ . Again since Σ is proper, the intersection of Σ_j with $\partial D = \gamma$ cannot be one point, which is a contradiction.

□

Combining Proposition 3 in Section 3 with Proposition 7, we obtain

Proposition 8. *Let $(M, \bar{g}, e^{-f} d\mu)$ be a simply connected complete smooth measure space. If a complete f -minimal hypersurface has finite weighted volume, then every component of Σ separates M into two components and thus is two-sided. Therefore Σ has a globally defined unit normal.*

We will take the same approach as in Colding-Minicozzi's paper [4] to prove Theorem 2, a smooth compactness theorem for complete f -minimal surfaces. First we recall a well known local singular compactness theorem for embedded minimal surfaces in a Riemannian 3-manifold.

Proposition 9. *(cf [4] Proposition 2.1) Given a point p in a Riemannian 3-manifold M , there exists a $R > 0$ such that : Let Σ_j be embedded minimal surfaces in $B_{2R}(p) \subset M$ with $\partial\Sigma_j \subset \partial B_{2R}(p)$. If each Σ_j has area at most V and genus at most g for some fixed V, g , then there is a finite collection of points x_k , a smooth embedded minimal surface $\Sigma \subset B_R(p)$ with $\partial\Sigma \subset \partial B_R(p)$ and a subsequence of $\{\Sigma_j\}$ that converges in $B_R(p)$ (with finite multiplicity) to Σ away from the set $\{x_k\}$.*

It is known that Σ is f -minimal with respect to metric \bar{g} if and only if Σ is minimal with the conformal metric $\tilde{g} = e^{-f} g$ (see Appendix). Using this fact and applying Proposition 9, we may prove a global singular compactness theorem for f -minimal surfaces.

Proposition 10. *Let M be a complete 3-manifold and $(M, \bar{g}, e^{-f} d\mu)$ a smooth metric measure space. Suppose that $\Sigma_i \subset M$ is a sequence of smooth complete embedded f -minimal surfaces with genus at most g , without boundary, and with weighted area at most V , i.e.*

$$(17) \quad \int_{\Sigma_i} e^{-f} d\sigma \leq V < \infty.$$

Then there is a subsequence, still denoted by Σ_i , a smooth embedded complete non-trivial f -minimal surface $\Sigma \subset M$ without boundary, and a locally finite collection of points $\mathcal{S} \subset \Sigma$ so that Σ_i converges smoothly (possibly with

multiplicity) to Σ off of \mathcal{S} . Moreover, Σ satisfies $\int_{\Sigma} e^{-f} d\sigma \leq V$ and is properly embedded.

Here a set $\mathcal{S} \subset M$ is said to be locally finite if $B_R(p) \cap \mathcal{S}$ is finite for every $p \in M$ and for all $R > 0$.

Proof. Consider the conformal metric $\tilde{g} = e^{-f}\bar{g}$ on M . For a point $p \in M$, let $\tilde{B}_{2R}(p) \subset M$ denote the ball in (M, \tilde{g}) of radius $2R$ centered at p . Then the area of $\tilde{B}_{2R}(p) \cap \Sigma$ satisfies

$$(18) \quad \widetilde{\text{Area}}(\tilde{B}_{2R}(p) \cap \Sigma) \leq \int_{\Sigma} d\tilde{\sigma} = \int_{\Sigma} e^{-f} d\sigma \leq V.$$

Also, it is clear that the genus of $\tilde{B}_{2R}(p) \cap \Sigma_j$ remains at most g . Then by Proposition 9, there exists a finite collection of points x_k , a smooth embedded minimal surface $\Sigma \subset \tilde{B}_R(p)$, with $\partial\Sigma \subset \partial\tilde{B}_R$ and a subsequence of $\{\Sigma_j\}$ that converges in $\tilde{B}_R(p)$ (with finite multiplicity) to Σ away from the set $\{x_k\}$.

Let $\{\tilde{B}_{R_i}(p_i)\}$ be a countable cover of (M, \tilde{g}) . On each $\tilde{B}_{2R_i}(p_i)$, applying the previous local convergence, and then passing to a diagonal subsequence, we obtain that there is a subsequence of Σ_i , still denoted by Σ_i , a smooth embedded minimal surface Σ (with respect to the metric \tilde{g}) without boundary, and a locally finite collection of points $\mathcal{S} \subset \Sigma$ so that Σ_i converges smoothly (possibly with multiplicity) to Σ off of \mathcal{S} .

Since Σ has no boundary, it is complete in the original metric \bar{g} . Thus we obtain that the smooth convergence of the subsequence to the smooth embedded complete f -minimal surface Σ off of \mathcal{S} .

By Corollary 2, Σ is nontrivial.

The convergence of Σ_i to Σ and (17) imply $\int_{\Sigma} e^{-f} d\sigma \leq V$. By Proposition 3, Σ is properly embedded.

□

We need to show that the convergence is smooth across the points in \mathcal{S} . To prove it, we need the following

Proposition 11. *Assume that the ambient manifold M in Proposition 10 is simply connected. If the convergence of the sequence $\{\Sigma_i\}$ has the multiplicity greater than one, then Σ is L_f -stable.*

Proof. By Proposition 8, we know that Σ_i and Σ are orientable. We may have two ways to prove the proposition. The first is to use the known fact on minimal surfaces. It is known that (cf [6] Appendix A) if the multiplicity of the convergence of a sequence of embedded orientable minimal surfaces in a simply connected 3-manifold is not one, then the limit minimal surface is stable. Under the conformal metric \tilde{g} , a sequence $\{\Sigma_i\}$ of minimal surfaces converges to a smooth embedded orientable minimal surface Σ and thus Σ is stable. Besides, the conclusion that Σ is stable with respect to the conformal metric \tilde{g} is equivalent to that Σ is L_f -stable under the original metric \bar{g} (see Appendix).

The second way is to prove directly. We may prove that L_f is the linearization of f -minimal equation by a similar proof to the one in [5, 8] Appendix A. By arguing as in Proposition 3.2 in [4, 8], we can find a smooth positive function u on Σ satisfying

$$(19) \quad L_f u = 0.$$

This implies that Σ is L_f -stable. □

Proof of theorem 2. By the assumption on $\overline{\text{Ric}}_f$ and Proposition 1, M has finite fundamental group. After passing to the universal covering, we may assume that M is simply connected. Given a sequence of smooth complete embedded f -minimal surfaces $\{\Sigma_i\}$ with genus g , $\partial\Sigma_i = \emptyset$, and the weighted area at most V , by Proposition 10, there is a subsequence, still denoted by $\{\Sigma_i\}$ so that it converges in the topology of smooth convergence on compact subsets to a smooth embedded complete f -minimal surface Σ away from a locally finite set $\mathcal{S} \subset \Sigma$ (possibly with multiplicity). Moreover, the limit surface $\Sigma \subset M$ is complete, properly embedded, $\int_{\Sigma} e^{-f} d\sigma \leq V$, has no boundary and has a well-defined unit normal ν . We also have the equivalent convergence under the conformal metric \bar{g} .

If \mathcal{S} is not empty, Allard's regularity theorem implies that the convergence has multiplicity greater than one. Then by Proposition 11, we conclude that Σ is L_f -stable. But Proposition 5 says that there is no such Σ . This

contradiction implies that \mathcal{S} must be empty. We complete the proof of the theorem. \square

Remark 1. For self-shrinkers, the condition that the scale-invariant uniform area bound exists (i.e. there is a uniform bound V_1 : $\text{Area}(B_R(x_0) \cap \Sigma) \leq V_1 R^2$ for all $x_0 \in \mathbb{R}^3$ and $R > 0$) implies that the uniform bound V of weighted area (i.e. $\int_{\Sigma} e^{-f} d\sigma < V$) exists (cf. the proof of Proposition 5). The converse is also true by the conclusion that the entropy of a self-shrinker can be achieved by $F_{0,1}$ for self-shrinkers with polynomial volume growth (see Section 7 of [5]). Therefore Theorem 2 generalizes the result of Colding-Minicozzi (Theorem 1) for self-shrinkers.

Remark 2. Combining Theorem 2 with the upper bound estimate of weighted area for closed embedded f -minimal surfaces of fixed genus in a complete 3-manifold with $\overline{\text{Ric}}_f \geq k > 0$, we may obtain the smooth compactness theorem for the space of closed embedded f -minimal surfaces of fixed topological type and with diameter bound. We discuss it in [2].

6. APPENDIX A

In this Appendix, we discuss the L_f -stability properties of f -submanifolds. With the same notations as in Section 2, let (M^m, \bar{g}) be an m -dimensional Riemannian manifold and $i : \Sigma^n \rightarrow M^m, n < m$, be an immersion. Let $\tilde{g} = e^{-\frac{2}{n}f} \bar{g}$ denote the new conformal metric on M . Therefore i may induce two isometric immersions of Σ : $(\Sigma, \bar{g}) \rightarrow (M, \bar{g})$ and $(\Sigma, \tilde{g}) \rightarrow (M, \tilde{g})$ respectively.

When (Σ, \tilde{g}) is minimal, it is well known that the second variation of the volume of (Σ, \tilde{g}) is given by

Proposition 12. (cf [6]) *Let (Σ, \tilde{g}) be minimal submanifold in (M, \tilde{g}) . If T is a normal compactly supported variational vector field on Σ (that is, $T = T^\perp$), then the second variational formula of the volume \tilde{V} of (Σ, \tilde{g}) is given by*

$$(20) \quad \left. \frac{d^2}{dt^2} \tilde{V}(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \langle T, JT \rangle_{\tilde{g}} d\tilde{\sigma},$$

where the stability operator (or Jacobi operator) J is defined on a normal vector field T to Σ by

$$(21) \quad JT = \Delta_{(\Sigma, \tilde{g})}^\perp T + \text{tr}_{(\Sigma, \tilde{g})}[\widetilde{Rm}(\cdot, T) \cdot]^\perp + \tilde{B}(T).$$

Here $\Delta_{(\Sigma, \tilde{g})}^\perp T = \sum_{i=1}^n (\nabla_{\tilde{e}_i}^\perp \nabla_{\tilde{e}_i}^\perp T - \nabla_{\nabla_{\tilde{e}_i}^\perp T}^\perp T)$ is the Laplacian determined by the normal connection ∇^\perp of (Σ, \tilde{g}) , \widetilde{Rm} is the curvature tensor on (M, \tilde{g}) , $\text{tr}_{(\Sigma, \tilde{g})}[\widetilde{Rm}(\cdot, T) \cdot]^\perp = \sum_{i=1}^n [\widetilde{Rm}(\tilde{e}_i, T) \tilde{e}_i]^\perp$, \tilde{A} denotes the second fundamental form of (Σ, \tilde{g}) , $\tilde{B}(T) = \sum_{i,j=1}^n \langle \tilde{A}(\tilde{e}_i, \tilde{e}_j), T \rangle \tilde{A}(\tilde{e}_i, \tilde{e}_j)$, and $\{\tilde{e}_i\}$, $i = 1, \dots, n$ is a local orthonormal base of (Σ, \tilde{g}) .

Recall that the weighted volume of (Σ, \bar{g}) is defined by

$$(22) \quad V_f(\Sigma) = \int_\Sigma e^{-f} d\sigma.$$

By a direct computation similar to that of (20), we may prove the second variational formula of the weighted volume of f -minimal submanifold (Σ, \bar{g}) .

Definition 5. For any normal vector field T on (Σ, \bar{g}) , the second order operator Δ_f^\perp is defined by

$$\begin{aligned} \Delta_f^\perp T &:= \Delta^\perp T - \text{tr}[\nabla f \otimes \nabla^\perp T(\cdot, \cdot)] \\ &= \sum_{i=1}^n (\nabla_{e_i}^\perp \nabla_{e_i}^\perp T - \nabla_{\nabla_{e_i}^\perp T}^\perp T) - \sum_{i=1}^n (e_i f)(\nabla_{e_i}^\perp T) \end{aligned}$$

The operator L_f on (Σ, \bar{g}) is defined by,

$$(23) \quad L_f T = \Delta_f^\perp T + R(T) + B(T) + F(T).$$

In the above, ∇^\perp denotes the normal connection of (Σ, \bar{g}) ; $\{e_i\}$, $i = 1, \dots, n$ is a local orthonormal base of (Σ, \bar{g}) ; $B(T) = \sum_{i,j=1}^n \langle A(e_i, e_j), T \rangle A(e_i, e_j)$, where A denotes the second fundamental form of (Σ, \bar{g}) ;

$R(T) = \text{tr}_{(\Sigma, \bar{g})}[\overline{Rm}(\cdot, T) \cdot]^\perp = \sum_{i=1}^n [\overline{Rm}(e_i, T) e_i]^\perp$, where \overline{Rm} denotes the Riemannian curvature tensor of (M, \bar{g}) ; and

$F(T) = [\bar{\nabla}^2 f(T)]^\perp = \sum_{\alpha=n+1}^m \bar{\nabla}^2 f(T, e_\alpha) e_\alpha$, where $\{e_\alpha\}$, $\alpha = n+1, \dots, m$ is a local orthonormal normal vector field on (Σ, \bar{g}) .

Proposition 13. *Let (Σ, \bar{g}) be an f -minimal in (M, \bar{g}) . If T is a normal compactly supported variational vector field on Σ (that is, $T = T^\perp$), then the second variational of the weighted volume of (Σ, \bar{g}) is given by*

$$(24) \quad \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \langle T, L_f T \rangle_{\bar{g}} e^{-f} d\sigma.$$

Proof. Let $\psi(\cdot, t), t \in (-\varepsilon, \varepsilon)$ be a compactly supported variation of Σ so that $T = d\psi(\frac{\partial}{\partial t})$ is the variational vector field, $\Sigma_t = \psi(\Sigma, t), \Sigma_0 = \Sigma$. Choose a normal coordinate system $\{x_1, \dots, x_n\}$ at a point $p \in \Sigma$. We can consider $\{x_1, \dots, x_n, t\}$ to be a coordinate system of $\Sigma \times (-\varepsilon, \varepsilon)$ near the point $(p, 0)$. Denote $e_i = d\psi(\frac{\partial}{\partial x_i})$ for $i = 1, \dots, n$. The induced metric on Σ_t from (M, \bar{g}) is given for $g_{ij} = \langle e_i, e_j \rangle$. Hence $g_{ij}(p, 0) = \delta_{ij}$ and $\nabla_{e_i} e_j(p, 0) = 0$. Denote by $d\sigma_t$ the volume element of Σ_t . Then $d\sigma_t = J(x, t)d\sigma_0$, where $d\sigma_0 = d\sigma$ and the function $J(x, t)$ is given by

$$J(x, t) = \frac{\sqrt{G(x, t)}}{\sqrt{G(x, 0)}},$$

with $G(x, t) = \det(g_{ij}(x, t))$. Denote by $d(\sigma_f)_t$ the weighted volume element of Σ_t . Then $d(\sigma_f)_t = J_f(x, t)d\sigma_0$, where $J_f(x, t) = J(x, t)e^{-f(x, t)}, f(x, t) = f(\psi(x, t))$.

Since $\frac{\partial J}{\partial t} = \sum_{i,j=1}^n g^{ij} \langle \bar{\nabla}_{e_i} T, e_j \rangle J$, $\frac{\partial J_f}{\partial t} = (\sum_{i,j=1}^n g^{ij} \langle \bar{\nabla}_{e_i} T, e_j \rangle - \langle \bar{\nabla} f, T \rangle) J_f$.

Note that T is a normal vector field. A direct computation gives, at $(p, 0)$

$$\begin{aligned} \left. \frac{\partial^2 J_f}{\partial^2 t} \right|_{t=0} = & \left[-2 \sum_{i,j=1}^n \langle A_{ij}, T \rangle^2 + \langle \bar{R}(e_i, T)T, e_i \rangle \right. \\ & + \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla}_T T, e_i \rangle + \sum_{i=1}^n \langle \bar{\nabla}_{e_i} T, \bar{\nabla}_{e_i} T \rangle \\ & - \bar{\nabla}^2 f(T, T) - \langle \bar{\nabla} f, \bar{\nabla}_T T \rangle \\ & \left. + \left(\sum_{i=1}^n \langle \bar{\nabla}_{e_i} T, e_i \rangle - \langle \bar{\nabla} f, T \rangle \right) \left(\sum_{j=1}^n \langle \bar{\nabla}_{e_j} T, e_j \rangle - \langle \bar{\nabla} f, T \rangle \right) \right] J_f. \end{aligned}$$

By

$$\begin{aligned}
\sum_{i=1}^n \langle \bar{\nabla}_{e_i} T, \bar{\nabla}_{e_i} T \rangle &= \sum_{i,j=1}^n \langle \bar{\nabla}_{e_i} T, e_j \rangle^2 + \sum_{i=1}^n \sum_{\alpha=n+1}^m \langle \bar{\nabla}_{e_i} T, e_\alpha \rangle^2 \\
&= \sum_{i,j=1}^n \langle A_{ij}, T \rangle^2 + \sum_{i=1}^n \langle \nabla_{e_i}^\perp T, \nabla_{e_i}^\perp T \rangle \\
&= |\langle A(\cdot, \cdot), T \rangle|^2 + |\nabla^\perp T|^2,
\end{aligned}$$

and $\sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla}_T T, e_i \rangle = \operatorname{div}(\bar{\nabla}_T T)^\top - \langle (\bar{\nabla}_T T)^\perp, \mathbf{H} \rangle$, we have that, at p

$$\begin{aligned}
\frac{\partial^2 J_f}{\partial t^2} \Big|_{t=0} &= \left[-|\langle A(\cdot, \cdot), T \rangle|^2 - \sum_{i=1}^n \langle \bar{R}(e_i, T) e_i, T \rangle + |\nabla^\perp T|^2 + \operatorname{div}(\bar{\nabla}_T T)^\top \right. \\
&\quad \left. - \langle (\bar{\nabla}_T T)^\perp, \vec{H} \rangle - \bar{\nabla}^2 f(T, T) - \langle \bar{\nabla} f, \bar{\nabla}_T T \rangle + \langle T, \mathbf{H}_f \rangle^2 \right] e^{-f}.
\end{aligned}$$

Using $\operatorname{div}(e^{-f}(\bar{\nabla}_T T)^\top) = e^{-f} \operatorname{div}(\bar{\nabla}_T T)^\top - e^{-f} \langle (\bar{\nabla}_T T)^\top, \nabla f \rangle$, we have at p

$$\begin{aligned}
(25) \quad & \frac{\partial^2 J_f}{\partial t^2} \Big|_{t=0} \\
&= \left[|\nabla^\perp T|^2 - |\langle A(\cdot, \cdot), T \rangle|^2 - \sum_{i=1}^n \langle \bar{R}(e_i, T) e_i, T \rangle - \bar{\nabla}^2 f(T, T) \right. \\
&\quad \left. - \langle (\bar{\nabla}_T T)^\perp, \mathbf{H}_f \rangle + \langle T, \mathbf{H}_f \rangle^2 \right] e^{-f} + \operatorname{div}(e^{-f}(\bar{\nabla}_T T)^\top).
\end{aligned}$$

Observe that the right-hand side of (25) is independent of the choice of coordinates. Hence (25) holds on Σ . By integrating (25) and using the fact that Σ is f -minimal (i.e., $\mathbf{H}_f = 0$), we obtain

$$\begin{aligned}
\frac{d^2}{dt^2} V_f(\Sigma_t) \Big|_{t=0} &= \int_\Sigma (|\nabla^\perp T|^2 - |\langle A(\cdot, \cdot), T \rangle|^2 - \langle R(T), T \rangle - \bar{\nabla}^2 f(T, T)) e^{-f} d\sigma \\
&= - \int_\Sigma \langle T, \Delta_f^\perp T + A(T) + R(T) + F(T) \rangle e^{-f} d\sigma \\
&= - \int_\Sigma \langle T, L_f T \rangle e^{-f} d\sigma.
\end{aligned}$$

Substituting $e^{-f} T$ for T in the identity $\int_\Sigma |\nabla^\perp T|^2 d\sigma = - \int_\Sigma \langle T, \Delta^\perp T \rangle d\sigma$, we have

$$\int_\Sigma |\nabla^\perp T|^2 e^{-f} d\sigma = - \int_\Sigma \langle T, \Delta_f^\perp T \rangle e^{-f} d\sigma.$$

Thus we have the second variational formula of the weighted volume of Σ

$$\begin{aligned} \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} &= - \int_{\Sigma} \langle T, \Delta_f^\perp T + A(T) + R(T) + F(T) \rangle e^{-f} d\sigma \\ &= - \int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma. \end{aligned}$$

□

Definition 6. *An f -minimal submanifold (Σ, \bar{g}) is called L_f -stable if the second variational of the weighted volume of Σ given by (24) is nonnegative for any normal compactly supported variational vector field T on Σ .*

Observe that for an f -minimal submanifold Σ and its normal compactly supported variation, it holds that $V_f(\Sigma_t) = \tilde{V}(\Sigma_t)$. Then

$$(26) \quad \left. \frac{d^2}{dt^2} \tilde{V}(\Sigma_t) \right|_{t=0} = \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0}.$$

By (20), (24), and (26), we have

$$(27) \quad \int_{\Sigma} \langle T, JT \rangle_{\bar{g}} e^{-f} d\tilde{\sigma} = \int_{\Sigma} \langle T, L_f T \rangle_{\bar{g}} e^{-f} d\sigma.$$

This implies that

$$(28) \quad \int_{\Sigma} e^{-\frac{2f}{n}} \langle T, JT \rangle_{\bar{g}} e^{-f} d\sigma = \int_{\Sigma} \langle T, L_f T \rangle_{\bar{g}} e^{-f} d\sigma.$$

By (28), the following equality holds.

Corollary 3. *For any normal variation field T on Σ ,*

$$JT = e^{\frac{2f}{n}} L_f T.$$

The operator L_f corresponds to a symmetric bilinear form $B_f(T, T)$ for the space of normal compactly supported field on Σ :

$$(29) \quad B_f(T, T) := - \int_{\Sigma} \langle T, L_f T \rangle_{\bar{g}} e^{-f} d\sigma.$$

We define the L_f -index, denoted by $L_f\text{-ind}$, of (Σ, \bar{g}) by the maximum of the dimensions of negative definite subspaces of B_f . Hence (Σ, \bar{g}) is L_f -stable if and only if its $L_f\text{-ind} = 0$.

On the other hand, for minimal (Σ, \tilde{g}) , it is well known that the stability operator J also defines a symmetric bilinear form $\tilde{B}(T, T)$,

$$(30) \quad \tilde{B}(T, T) := - \int_{\Sigma} \langle T, JT \rangle_{\tilde{g}} d\tilde{\sigma}.$$

There are also the concepts of index and stability of (Σ, \tilde{g}) . In particular, (Σ, \tilde{g}) is stable if and only if the index $\text{ind}(\Sigma, \tilde{g}) = 0$. Since $B_f(T, T) = \tilde{B}(T, T)$, it holds that

Proposition 14. *L_f -ind of (Σ, \bar{g}) is equal to the index of (Σ, \tilde{g}) . In particular, (Σ, \bar{g}) is L_f -stable if and only if (Σ, \tilde{g}) is stable in (M, \tilde{g}) .*

Now if Σ is a two-sided hypersurface, that is, there is a globally-defined unit normal ν on (Σ, \bar{g}) . Take $T = \varphi\nu$. Then the second variation (24) implies that

Proposition 15. *Let Σ be a two-sided f -minimal hypersurface in (M^{n+1}, \bar{g}) . If φ is a compactly supported smooth function on Σ , then the second variation of the weighted volume of (Σ, \bar{g}) is given by*

$$(31) \quad \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \varphi L_f(\varphi) e^{-f} d\sigma$$

where ν denotes the unit normal of (Σ, \bar{g}) and the operator L_f is defined by $L_f = \Delta_f + |A|_{\bar{g}}^2 + \overline{\text{Ric}}_f(\nu, \nu)$.

Definition 7. *The operator $L_f = \Delta_f + |A|_{\bar{g}}^2 + \overline{\text{Ric}}_f(\nu, \nu)$ is called L_f -stability operator of hypersurface (Σ, \bar{g}) .*

A bilinear form on space $C_0^\infty(\Sigma)$ of compactly supported smooth functions on Σ is defined by

$$(32) \quad \begin{aligned} B_f(\varphi, \varphi) &:= - \int_{\Sigma} \varphi L_f \varphi e^{-f} d\sigma \\ &= \int_{\Sigma} [|\nabla \varphi|^2 - (|A|_{\bar{g}}^2 + \overline{\text{Ric}}_f(\nu, \nu)) \varphi^2] e^{-f} d\sigma. \end{aligned}$$

The L_f -index, denoted by $L_f\text{-ind}$, of (Σ, \bar{g}) is defined to be the maximum of the dimensions of negative definite subspaces of B_f . Hence (Σ, \bar{g}) is L_f -stable if and only if $L_f\text{-ind} = 0$. Clearly the definition of L_f -index is equivalent to the corresponding definition using the variational field T before.

Also, for minimal hypersurface $i : (\Sigma, \tilde{g}) \rightarrow (M^{n+1}, \tilde{g})$, it is well known that if ψ is a compactly supported smooth function on Σ , then the second

variational of the volume \tilde{V} of $(\Sigma, i^*\tilde{g})$ is given by

$$(33) \quad \left. \frac{d^2}{dt^2} \tilde{V}(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \psi J(\psi) d\tilde{\sigma},$$

where \tilde{A} denotes the second fundamental form of (Σ, \tilde{g}) , $\tilde{\nu}$ denotes the unit normal of (Σ, \tilde{g}) , and $J = \Delta_{\tilde{g}} + |\tilde{A}|_{\tilde{g}}^2 + \widetilde{\text{Ric}}(\tilde{\nu}, \tilde{\nu})$ is stability operator (or the Jacobi operator) of (Σ, \tilde{g}) .

It holds that, from (28),

Proposition 16. *Let (Σ^n, g) be an f -minimal hypersurface immersed in (M, \bar{g}) . Then for all $\varphi \in C_0^\infty(\Sigma)$,*

$$(34) \quad \int_{\Sigma} (e^{-\frac{f}{n}} \varphi) J(e^{-\frac{f}{n}} \varphi) e^{-f} d\sigma = \int_{\Sigma} \varphi L_f(\varphi) e^{-f} d\sigma.$$

Corollary 4. *For $\varphi \in C^\infty(\Sigma)$, $J(e^{-\frac{f}{n}} \varphi) = e^{\frac{f}{n}} L_f(\varphi)$.*

Corollary 5. *L_f -ind of (Σ, \bar{g}) is equal to the index of (Σ, \tilde{g}) . In particular, (Σ, \bar{g}) is L_f -stable if and only if (Σ, \tilde{g}) is stable in (M, \tilde{g}) .*

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